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**Mathematical Problems Related to a Phase Transition**

**Final Report**

Harumi Hattori

September 15, 1998

U.S. Army Research Office

West Virginia University

## A. Statement of the Problem Studied

There are four areas that I studied. First, I studied the viscoelasticity of memory type. The typical system is given by

$$
(0.1) \qquad \begin{aligned} v_t &= u_x \\ u_t &= \sigma(v)_x + \textstyle\int_{-\infty}^{t} a'(t-\tau)\eta(v)_x \, d\tau \end{aligned}
$$

In (0.1), $u$ and $v$ are velocity and strain, respectively. For $\sigma$ and $\eta$ we require that $\sigma' > 0$ and $\eta' > 0$. We also require that $\chi(v) \equiv \sigma(v) - a(0)\eta(v)$ satisfies $\chi' > 0$. The condition $\chi' > 0$ means in particular that the equilibrium stress modulus is positive. An interesting aspect is that the memory term induces a dissipation. I studied the stability of rarefaction waves in [1,2]. With Kawashima we published three papers. In [3] the existence of shock profiles were studied. We discussed the necessary and sufficient conditions for the existence of smooth monotone shock profiles for viscoelastic materials with memory. We also discussed the uniqueness. We considered both convex and nonconvex constitutive relations. In the case of nonconvex constitutive relations, we include a degenerate case where the speed of the shock profile is equal to the speed of the equilibrium characteristics at one of the end states. This was not discussed in the previous literatures. In [4], we studied the stability of smooth monotone travelling wave solutions for viscoelastic materials with memory. It is known that a smooth monotone travelling wave solution exists for (0.1) if the end states are close and satisfy the Rankine-Hugoniot condition. For such a travelling wave, we shall show that if the initial data are close to a travelling wave solution, the solutions to (0.1) will approach the travelling wave solution in sup norm as the time goes to infinity. For the constitutive relations we shall discuss two cases, the convex and nonconvex cases. We summarized the recent results concerning the viscoelasticity of memory type in [5].

Second, I discussed the Korteweg materials where the effect of capillarity was included in the model. Dunn and Serrin proposed the interstitial working term and modified the system of compressible fluids based on the Korteweg theory of capillarity. This term was introduced to overcome the difficulty that the higher order terms of density is not compatible with the classical theory of thermodynamics. The system we studied is

$$(0.2) \qquad \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$(0.3) \qquad \rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot (\mathbf{T} + \mathbf{V}),$$

$$(0.4) \qquad \rho \varepsilon_t + \rho \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\varepsilon} = (\mathbf{T} + \mathbf{V}) \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \nabla \cdot \mathbf{z},$$

$$(0.5) \qquad \rho(\dot{\psi} + \eta \, \dot{\theta}) - (\mathbf{T} + \mathbf{V}) \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{z} + \frac{\mathbf{q} \cdot \nabla \theta}{\theta} \leq 0,$$

where $\dot{f} = f_t + \mathbf{u} \cdot \nabla f$ and

1. $\rho$ is the density,

2. $\mathbf{u}$ is the velocity,

3. $\theta$ is the absolute temperature,

4. $\varepsilon$ is the specific internal energy per unit mass,

5. $\eta$ is the specific entropy per unit mass,

6. $\psi = \varepsilon - \theta\eta$ is the Helmholtz free energy,

7. $\mathbf{T}$ is the Cauchy stress tensor,

8. $\mathbf{V}$ is the viscosity tensor,

9. $\mathbf{q}$ is the heat flux vector.

For the elastic materials of Korteweg type, using the Clausius-Duhem inequality they have shown that the following forms of $\mathbf{z}$ and $\mathbf{T}$

$$\mathbf{z} = \rho\,\dot{\rho}\,\psi_{\mathbf{d}}, \tag{0.6}$$

$$\mathbf{T} = (-\rho^2\psi_\rho + \rho\nabla\cdot(\rho\psi_{\mathbf{d}}))\mathbf{I} - \rho\mathbf{d}\otimes\psi_{\mathbf{d}} \tag{0.7}$$

are compatible with (0.5). Here, $\rho^2\psi_\rho(\rho,\theta,\mathbf{0}) = p(\rho,\theta)$ is the pressure and $\mathbf{d} = \nabla\rho$. They also have observed that the classical forms of viscosity and heat conductivity tensors are compatible. In what follows, we use the viscosity tensor and the heat flux vector given, respectively, by

$$\mathbf{V} = \mu\{(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T - \tfrac{2}{3}(\nabla\cdot\mathbf{u})\mathbf{I}\},$$
$$\mathbf{q} = -\nabla\theta.$$

For the Helmholtz free energy, we assume that it is given by

$$\psi(\rho,\theta,\mathbf{d}) = \sigma(\rho,\theta) + \lambda(\rho,\theta)(\mathbf{d}\cdot\mathbf{d}), \qquad \lambda > 0,$$

where $\sigma$ and $\lambda$ are smooth functions of their arguments. In this case

$$\eta = -\sigma_\theta - \lambda_\theta(\mathbf{d}\cdot\mathbf{d}),$$

$$\varepsilon = \sigma - \theta\sigma_\theta + (\lambda - \theta\lambda_\theta)\mathbf{d}\cdot\mathbf{d}.$$

I collaborated with Li and published three papers. First, in [6,7] we studied the local existence and then in [8] we discussed the global existence. In [9] we considered the case where the energy equation is included.

Third, I studied the hyperbolic-elliptic mixed type system related to phase transition. The system is given by

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - f(v)_x &= 0, \end{aligned} \tag{0.8}$$

where $f$ is given as in Figure 1.

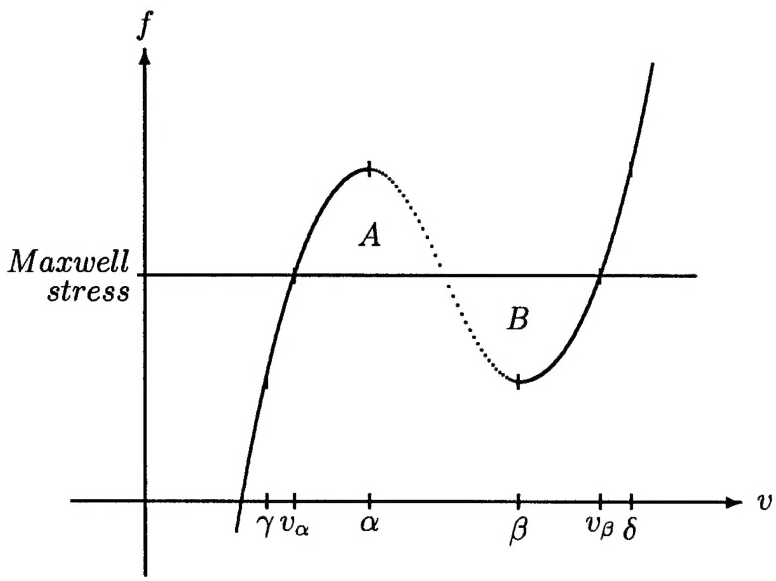


Figure 1

In [10] we discuss the Riemann Problem and existence of weak solutions to a dynamic phase transition problem. To estimate the wave strength of outgoing waves containing the phase boundary, we obtain the reflection and transmission coefficients for phase boundary. Then, employing the entropy rate admissibility criterion and the initiation criterion, we prove the existence of global weak solutions via Glimm scheme. We also studied the various properties of entropy rate admissibility criterion.

I also studied the hydrodynamic models of semiconductor. The system is given by

$$n_t + (nv)_x = 0, \tag{0.9}$$

$$(nv)_t + (nv^2 + p)_x = -nF - \frac{nv}{\tau_p}, \tag{0.10}$$

$$F_x = -n - n_d(x), \tag{0.11}$$

In [11,12] I studied the stability of steady state solutions and in [13] we considered the effect of heat conduction. We study the case where the doping profile is close to a positive constant and depends on the spacial variable $x$. We have shown that the steady state solution is asymptotically stable with respect to small perturbations in an appropriate Sobolev space.

## B. Summary of the Most Important Results

The main findings are summarized as follows:

1. The role of memory term in the viscoelasticity with memory became clearer in the context of the stability of rarefaction waves and the traveling waves. The memory term induces a subtle dissipation. This effect was examined in the above context.

2. The effects of capillarity term was investigated. The capillarity effect is important where there is a rapid change of density. One example is a phase transition. We have shown the well-posedness for a fluid dynamic model containing capillary effects.

3. The entropy rate admissibility criterion was used to show the existence of weak solution and also the various properties of the entropy rate admissibility criterion was studied. Through this study the relation between the entropy rate admissibility criterion and the entropy condition became clearer.

4. The stability of steady state solutions is interesting in the sense that the system contains an elliptic equation for the potential. This equation makes the stability proof more interesting. This is another type of elliptic-hyperbolic (or parabolic) mixed type problem. This type of mixed problem has not been studied. We see the effect of elliptic equation in the time dependent problems.

## C. List of All Publications


1. Nonlinear Stability of Rarefaction Waves for a Viscoelastic Material with Memory, Transactions of American Mathematical Society, 342 (1994), 645-669.

2. Stability of Rarefaction Waves for a Viscoelasticity, Journal of Differential Equations 111 (1994), 1-26.

3. (With S. Kawashima) Existence of shock profiles for materials with memory, SIAM J. on Mathematical Analysis 26 (1995), 1130-1142.

4. (With S. Kawashima) Nonlinear stability of travelling wave solutions for viscoelastic materials with memory, Journal of Differential Equations 127 (1996), 174-196.

5. (With S. Kawashima) Smooth Shock Profiles in Viscoelasticity with Memory, Studies in Advanced Mathematics 3 (1997), 271-281.

6. (With D. Li) Solutions for two dimensional system for materials of Korteweg type, SIAM J. on Mathematical Analysis, 25 (1994), 85-98.

7. (With D. Li) A System for Materials of Korteweg Type in Two Space Variables, Transactions of the Eleventh Army Conference on Applied Mathematics and Computing 1994.

8. (With D. Li) Global solutions of high dimensional system for Korteweg materials, Mathematical Analysis and Applications 198 (1996), 84-97.

9. (With D. Li) The Existence of Global Solutions to a Fluid Dynamic Model for Materials of Korteweg Type, Journal of Partial Differential Equations 9 (1996), 323-342.

10. The Riemann problem and the Existence of Weak Solutions to a System of Mixed-Type in Dynamic Phase Transition, Journal of Differential Equations, 146 (1998), 287-319.

11. Stability of Steady State Solutions for a Hydrodynamic Model for Semiconductors, in Smart Structures and Materials 1995: Mathematics and Control in Smart Structures, Vasundara V. Varadan, Editor, Proc. SPIE 2442, 493-502 (1995).

12. Stability and Instability of Steady State Solutions for a Hydrodynamic Model for Semiconductors, Proceedings of Royal Society of Edinburgh, 127A (1997), 781-796.

13. (With C. Zhu) Asymptotic Behavior of the Solution to a Nonisentropic Hydrodynamic Model of Semiconductors, Journal of Differential Equations, 144 (1998), 353-389.

**D. Scientific Personnel**

Chen Zhu: completed his Ph.D. program in December '97.